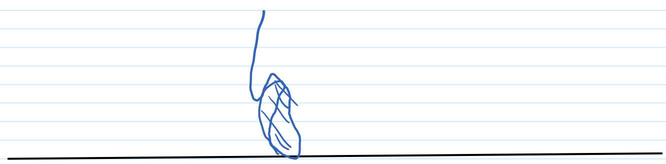
Chordal Loewner Evolution

$\mathbb{H} = \{\operatorname{Im} z > 0\}$ - the upper half-plane.

Def. $A \subset \mathbb{H}$ - compact hull if 1) $\bar{A}$ - compact,
2) $A = \bar{A} \cap \mathbb{H}$.
3) $\mathbb{H} \setminus A$ - simply-connected connected domain

Main example Curve γ from 0 to ∞ generates family of hulls.

$\mathbb{H} \setminus A_t$ = unbounded component of $\mathbb{H} \setminus \gamma([0,t])$.



Fact. $\exists$ unique map $g_A : \mathbb{H} \setminus A \to \mathbb{H}$: $\lim_{z \to \infty} (g_A(z) - z) = 0$

Hydrodynamic normalization

Pf By Riemann, $\exists g : \mathbb{H} \setminus A \to \mathbb{H}$, with $g(\infty) = \infty$. Extendable to $\tilde{g} : \hat{\mathbb{C}} \setminus Clos(A \cup \bar{A})$, $\to \hat{\mathbb{C}}$, mapping $\mathbb{R} \to \mathbb{R}$. & expansion at ∞, $\tilde{g}(z) = Az + B + \ldots$, where $A, B \in \mathbb{R}$.
Normalize ∎

Def. (Half-plane capacity)
$hcap(A) = \lim_{z \to \infty} z(g_A(z) - z)$.

Lemma If $r > 0$, $hcap(rA) = r^2 hcap(A)$.

Pf $g_{rA}(z) = r\, g_A(\frac{z}{r})$ ∎

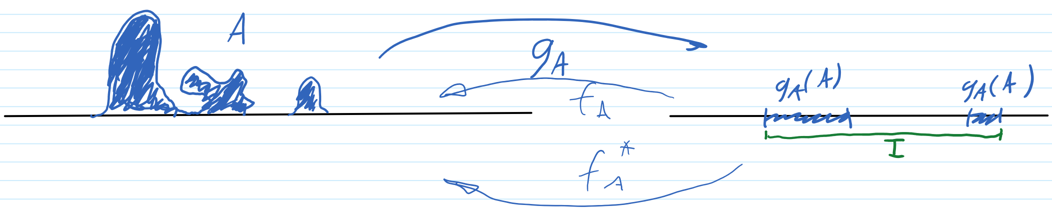
Examples 1) $A=\overline{\mathbb{D}}\cap\mathbb{H}$, $g_A(z)=z+\frac{1}{z}$, $\text{hcap}\, A=1$.
2) $A=[0,i]$, then $g_A(z)=\sqrt{z^2+1}=z+\frac{1}{2z}+\dots \Rightarrow \text{hcap}\, A=\frac{1}{2}$.

Lemma $\text{hcap}\, A\geq 0$.

Pf: Let $v(z):=\operatorname{Im}(z-g_A(z))$.
Then $\lim_{z\to\infty} v(z)=0$, $v(z)\geq 0$ on $\mathbb{R}\cup\partial A$. By maximum principle, $v(z)>0$ on $\mathbb{H}$.
$\text{hcap}\, A=\lim_{z\to\infty} z(z-g_A z)=-\lim_{y\to\infty} iy\, i\, v(iy)\geq 0$ ∎

$f_A:=g_A^{-1}$

Assume: A - locally connected



I - minimal interval containing $g_A(\overline{A})$. Extend f to I by local connectedness.
Extend to f^* on $\mathbb{C}\setminus I$ by reflexion.

By Cauchy, $f^*(w)=\frac{1}{2\pi i}\int_{|z|=R}\frac{f^*(z)}{z-w}\,dz+\frac{1}{2\pi i}\int_I\frac{\overbrace{f_A(x)-\overline{f_A(x)}}^{2\operatorname{Im} f_A(x)\, i}}{x-w}\,dx$,
if $R>|w|$

Since at ∞, $f^*(z)=z-\text{hcap}\, A\cdot\frac{1}{z}+\dots$
we get

$$\lim_{R\to\infty}\int_{|z|=R}\frac{f^*(z)}{z-w}\,dz=\lim_{R\to\infty}\int_{|z|=R}\frac{z}{z-w}\,dz=2\pi i w.$$

Thus $f^*(w)-w=\frac{1}{\pi}\int_I\frac{\operatorname{Im} f_A(x)}{x-w}\,dx$. Multiply by w, take $w\to\infty$,

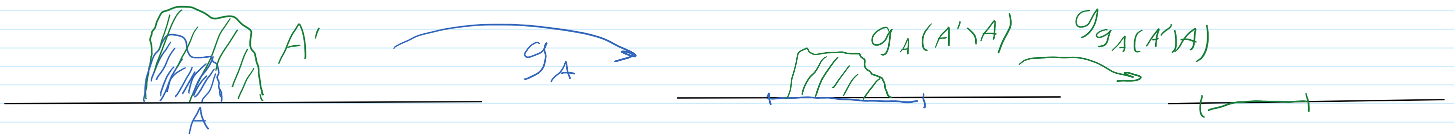
$$\boxed{\text{hcap}\, A=\frac{1}{\pi}\int_I \operatorname{Im} f_A(x)\,dx}$$

In particular, if $A \neq \emptyset$, $\mathrm{hcap}\, A > 0$; if A is locally connected.

Lemma. $A \subset A'$ - compact hulls. Then
$\mathrm{hcap}\, A' = \mathrm{hcap}\, A + \mathrm{hcap}\, g_A(A' \setminus A)$.

As a consequence: $\mathrm{hcap}\, A' \geq \mathrm{hcap}\, A$.

Pf. $g_{A'} = g_{g_A(A' \setminus A)} \circ g_A$, expand at ∞ ∎



Lemma. If $A \neq \emptyset$ - compact hull $\Rightarrow$ $\mathrm{hcap}\, A > 0$.
Pf. $\exists A' \subset A$ - locally connected, $\neq \emptyset$ ∎

Caratheodory convergence: $A \cup \bar{A} \cup \bar{I}$ converge in the usual Caratheodory sense w.r.t. ∞.

Class $\mathcal{P}$: $\mathrm{Re}\, p > 0$, $\lim_{z \to \infty} z p(z) = 1$.

Herglotz representation: $p(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z}$, $\mathrm{supp}\, \mu = \overline{\{t : \lim_{y \to 0+} p(t + iy) > 0\}}$.

Chordal L.C.: $f_t : \mathbb{H} \to \Omega_t := \mathbb{H} \setminus A_t$ where A_t - continuously growing family of compact hulls. Normalized L.C.: $\mathrm{hcap}\, A_t = 2t$.

Löwner equations:

$$\frac{\partial f_t}{\partial t} = -2 f_t'(z) \int_{\mathbb{R}} \frac{d\mu_t(x)}{z - x}, \quad f_0(z) \equiv z. \qquad g_t := f_t^{-1}.$$

$$\frac{\partial g_t(z)}{\partial t} = 2 \int \frac{d\mu_t(x)}{g_t(z) - x}, \quad g_0(z) \equiv z.$$

For curves:

$$\frac{\partial g_t}{\partial t}(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad \lambda(t) = g_t(\gamma(t)) \text{ - continuous driving function}$$

An analogue of Pommerenke holds, with a crosscut separating from ∞.

Bonus property: (Brownian scaling)

Driving function for $\underbrace{\alpha A_t}$ is $\alpha\lambda(\alpha^{-2}t)$ (since $g_{\alpha A_t}(z) = \alpha\, g_{A_t}(\frac{z}{\alpha})$).

Corollary: $B\sqrt{t}$ generates straight line segment. For $\forall B \in \mathbb{R}$.